IMSC 2048 Midterm

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Excercise 1. *Below is a partial character table for a group G of order 12, with 4 conjugacy classes. The numbers in the top row are the sizes of the conjugacy classes.*

	(1)	(3)	(4)	(4)
	e	a	b	c
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1			
χ_4				

where $\omega = e^{2\pi\sqrt{-1}/3}$.

(a) Complete the character table by determining the missing entries. Please write down the orthogonality relations or formulas you use to determine the missing entries.

(b) Determine all normal subgroups of G from the character table.

Solution. (a) Let the dimensions of the irreducible representations be d_1, d_2, d_3, d_4. Since χ_1 and χ_2 are characters of 1-dimensional representations, $d_1 = d_2 = 1$. The sum of squares of the dimensions is equal to the order of the group $|G| = 12$, so:

$$1^2 + 1^2 + d_3^2 + d_4^2 = 12 \implies d_3^2 + d_4^2 = 10.$$

Since d_3, d_4 are positive integers, the only solution is $d_3 = 1$ and $d_4 = 3$ (or vice versa; we set $d_3 = 1$ and $d_4 = 3$). Thus $\chi_3(e) = 1$ and $\chi_4(e) = 3$.

For χ_3, since it is a 1-dimensional character, its values are roots of unity. The character χ_2 is also 1-dimensional, and we can define χ_3 by noting that the square of χ_2 or complex conjugate must also be an irreducible character. Since $(\chi_2)^2(b) = \overline{\chi_2(b)} = \overline{\omega} = \omega^2$ and $\overline{\chi_2(c)} = \overline{\omega^2} = \omega$, we have $\overline{\chi_2} = (1, 1, \omega^2, \omega)$, which is orthogonal to χ_1 and χ_2. Thus, we can set $\chi_3 = (1, 1, \omega^2, \omega)$.

To find χ_4, we use the column orthogonality relation: $\sum_i \chi_i(g)\overline{\chi_i(h)} = 0$ for $g \neq h$. Taking $g = a$ and $h = e$:

$$1(1) + 1(1) + 1(1) + \chi_4(a)(3) = 0 \implies 3\chi_4(a) = -3 \implies \chi_4(a) = -1.$$

Taking $g = b$ and $h = e$:

$$1(1) + \omega(1) + \omega^2(1) + \chi_4(b)(3) = 0 \implies 1 + \omega + \omega^2 + 3\chi_4(b) = 0.$$

Since $1 + \omega + \omega^2 = 0$, we get $\chi_4(b) = 0$. Taking $g = c$ and $h = e$:

$$1(1)+\omega^2(1)+\omega(1)+\chi_4(c)(3) = 0 \implies 1+\omega^2+\omega+3\chi_4(c) = 0 \implies \chi_4(c) = 0.$$

Thus, the completed character table is:

	(1) e	(3) a	(4) b	(4) c
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

(b) The normal subgroups correspond to the intersections of the kernels of the irreducible representations. The kernel of χ_i is $\ker(\chi_i) = \{g \in G \mid \chi_i(g) = \chi_i(e)\}$. From the table: - $\ker(\chi_1) = G$ - $\ker(\chi_2) = (1) \cup (3) = \{e\} \cup [a]$ (order $1+3 = 4$) - $\ker(\chi_3) = (1) \cup (3) = \{e\} \cup [a]$ (order 4) - $\ker(\chi_4) = (1) = \{e\}$ (order 1)

Since any normal subgroup is an intersection of some $\ker(\chi_i)$, the normal subgroups of G are G, the subgroup of order 4 namely $\{e\} \cup [a]$, and the trivial subgroup $\{e\}$. □

Excercise 2. *Let* $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix}$.

(a) Compute $A^T A$ and find its eigenvalues and eigenvectors.

(b) Find the singular values $\sigma_1 \geq \sigma_2 > 0$ and write the SVD decomposition of $A = QDP^T$. (You only need to write the first two columns of Q, D, and P.)

Solution. (a) Compute $A^T A$:

$$A^T A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0+1+1 & 0+0-1 \\ 0+0-1 & 1+0+1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

The trace is 4 and the determinant is $4 - 1 = 3$. The characteristic polynomial is $x^2 - 4x + 3 = (x-3)(x-1)$. The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 1$.

For $\lambda_1 = 3$:

$$\begin{pmatrix} 2-3 & -1 \\ -1 & 2-3 \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} \implies v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\lambda_2 = 1$:

$$\begin{pmatrix} 2-1 & -1 \\ -1 & 2-1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \implies v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

(b) The singular values are $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{1} = 1$. So $D = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$, and $P = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

The first two columns of Q are $u_i = \frac{1}{\sigma_i} A v_i$:

$$u_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}.$$

$$u_2 = \frac{1}{1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

To complete Q, we find a unit matrix column u_3 orthogonal to u_1 and u_2. Solving $x_1 + x_2 = 0$ and $-x_1 + x_2 + 2x_3 = 0$ gives $x_2 = -x_1$ and $2x_3 = 2x_1 \implies x_3 = x_1$. So a unit vector is $u_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$.

The full SVD is:

$$A = QDP^T = \begin{pmatrix} -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \\ 2/\sqrt{6} & 0 & 1/\sqrt{3} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right)^T.$$

□

Excercise 3. *Let $\rho \colon G \to \mathrm{GL}(V)$ be a finite dimensional representation of a finite group G over $\mathbb{C}$. If H is a subgroup of G, then the restriction $\rho|_H$ is a representation of H. Assume that ρ is the regular representation of G. Show that the restriction $\rho|_H$ is isomorphic to a direct sum of $[G : H]$ copies of the regular representation of H.*

Solution. Use character theory: we need the following result from homework, a representation ρ of G is isomorphic to a direct sum of n copies of the regular representation of G if and only if the character of ρ is nonzero only at the identity element and takes the value $n|G|$ there.

Or we can use the following result. Two representations ρ_1 and ρ_2 of G are isomorphic if and only if their characters χ_1 and χ_2 are equal. The character of a representation restricted to a subgroup is the restriction of the character to that subgroup. Then if $\chi_\rho = n\chi_{reg}$, we have $\rho \cong \rho_{reg}^{\oplus n}$.

The character of the regular representation of G is $\chi(g) = |G|$ if $g = e$ and 0 otherwise. When we restrict to H, the character of $\rho|_H$ is $\chi|_H(h) = |G|$ if $h = e$ and 0 otherwise. The regular representation of H has character $\psi(h) = |H|$ if $h = e$ and 0 otherwise. Since $\chi|_H = [G : H]\psi$, it follows that $\rho|_H \cong [G : H] \cdot \mathrm{Reg}(H)$, where $\mathrm{Reg}(H)$ is the regular representation of H. □

Excercise 4. *Let W be the real vector space of Hermitian 2×2 matrices*

$$W = \{A \in M_2(\mathbb{C}) \mid A^* = A\}, \qquad A^* = \overline{A^T}.$$

(a) Prove that the rule $P \cdot A = PAP^*$ defines a linear action of $\mathrm{SL}_2(\mathbb{C})$ on W.

(b) Prove that the function $\langle A, A' \rangle = \det(A + A') - \det(A) - \det(A')$ is a symmetricbilinear form on W, and that its signature is $(3, 1)$.

(c) Use (a) and (b) to define a group homomorphism $\mathrm{SL}_2(\mathbb{C}) \to O_{3,1}$ whose kernel is $\{\pm I\}$.

Solution. (a) For any $A \in W$, $(P \cdot A)^* = (PAP^*)^* = (P^*)^*A^*P^* = PAP^* = P \cdot A$, so $P \cdot A \in W$. The map $A \mapsto PAP^*$ is clearly linear in A. Furthermore, $I \cdot A = IAI^* = A$, and for $P_1, P_2 \in \mathrm{SL}_2(\mathbb{C})$,

$$(P_1P_2) \cdot A = P_1P_2A(P_1P_2)^* = P_1P_2AP_2^*P_1^* = P_1(P_2AP_2^*)P_1^* = P_1 \cdot (P_2 \cdot A).$$

Thus, this defines a linear action of $\mathrm{SL}_2(\mathbb{C})$ on W.

(b) Note that the determinant is a quadratic form in the entries of A, so its polarization $\det(A + A') - \det(A) - \det(A')$ defines a symmetric bilinear form. Any matrix $A \in W$ can be written as $A = \begin{pmatrix} x+t & y+iz \\ y-iz & x-t \end{pmatrix}$ for $x, y, z, t \in \mathbb{R}$. Then $\det(A) = (x+t)(x-t) - (y^2+z^2) = x^2 - t^2 - y^2 - z^2$. The quadratic form associated with the bilinear form is $Q(A) = \frac{1}{2}\langle A, A \rangle = \det(2A) - 2\det(A) = 2\det(A) = 2x^2 - 2t^2 - 2y^2 - 2z^2$. This clearly has one positive square (from x) and three negative squares (from t, y, z). In either convention, the signature is $(3, 1)$ or $(1, 3)$.

(c) For $P \in \mathrm{SL}_2(\mathbb{C})$, the action preserves the quadratic form:

$$Q(P \cdot A) = 2\det(PAP^*) = 2\det(P)\det(A)\det(P^*) = 2 \cdot 1 \cdot \det(A) \cdot 1 = Q(A).$$

Since it preserves the quadratic form, it preserves the associated bilinear form. Thus, each P induces an orthogonal transformation on $(W, \langle \cdot, \cdot \rangle) \cong \mathbb{R}^{3,1}$. This defines a homomorphism $\phi : \mathrm{SL}_2(\mathbb{C}) \to O_{3,1}$. The kernel consists of P such that $PAP^* = A$ for all $A \in W$. Setting $A = I$ gives $PP^* = I$, so $P^* = P^{-1}$. Then $PAP^{-1} = A$ implies P commutes with all matrices in W. Since W spans $M_2(\mathbb{C})$ over $\mathbb{C}$, P must be a scalar matrix $P = cI$. The condition $\det(P) = 1$ implies $c^2 = 1$, so $c = \pm 1$. Thus, the kernel is $\{\pm I\}$. □

Excercise 5. *Let V be the real vector space of Hermitian matrices in $M_n(\mathbb{C})$*

$$V = \{A \in M_n(\mathbb{C}) \mid A = \overline{A}^T\}$$

1. *Prove that $A - \sqrt{-1}I$ is invertible for every $A \in V$.*

2. *Define the map $f\colon V \to \mathrm{GL}_n(\mathbb{C})$ by $f(A) = (A + \sqrt{-1}I)(A - \sqrt{-1}I)^{-1}$. Show that $f(A)$ is unitary for every $A \in V$.*

3. *Give a characterization of the image of $f\colon V \to \mathrm{U}(n)$ in terms of eigenvalues, that is, which matrices can be written in the form $(A+\sqrt{-1}I)(A-\sqrt{-1}I)^{-1}$ for some $A \in V$.*

4. *Is the map f injective? Justify your answer.*

Proof. **1. Invertibility of $A - \sqrt{-1}I$.** Let $A \in V$ be Hermitian, i.e., $A = A^*$. For any eigenvalue $\lambda \in \mathbb{C}$ of A with eigenvector $v \neq 0$, we have $Av = \lambda v$. Taking the conjugate transpose and using $A = A^*$ gives $\bar{\lambda} v^* = v^* A$, so λ is real. Hence all eigenvalues of A are real.

Now $A - \sqrt{-1}I$ has eigenvalues $\lambda - \sqrt{-1}$ with $\lambda \in \mathbb{R}$. Since $\lambda - \sqrt{-1} \neq 0$ (its imaginary part is -1), the matrix has no zero eigenvalues and is therefore invertible.

2. Unitarity of $f(A)$. For $A \in V$, define $f(A) = (A+\sqrt{-1}I)(A-\sqrt{-1}I)^{-1}$. Note that $A + \sqrt{-1}I$ and $A - \sqrt{-1}I$ commute because they are polynomials in A. Compute the conjugate transpose:

$$f(A)^* = \big((A + \sqrt{-1}I)(A - \sqrt{-1}I)^{-1}\big)^* = \big((A - \sqrt{-1}I)^{-1}\big)^*(A + \sqrt{-1}I)^*.$$

Since A is Hermitian, $(A-\sqrt{-1}I)^* = A+\sqrt{-1}I$ and $(A+\sqrt{-1}I)^* = A-\sqrt{-1}I$. Hence $\big((A - \sqrt{-1}I)^{-1}\big)^* = (A + \sqrt{-1}I)^{-1}$. Therefore

$$f(A)^* = (A + \sqrt{-1}I)^{-1}(A - \sqrt{-1}I).$$

Now multiply:

$$f(A)f(A)^* = (A + \sqrt{-1}I)(A - \sqrt{-1}I)^{-1}(A + \sqrt{-1}I)^{-1}(A - \sqrt{-1}I).$$

All factors are polynomials in A, so they commute. Rearranging gives

$$f(A)f(A)^* = (A + \sqrt{-1}I)(A + \sqrt{-1}I)^{-1}(A - \sqrt{-1}I)^{-1}(A - \sqrt{-1}I) = I \cdot I = I.$$

Similarly, $f(A)^*f(A) = I$. Thus $f(A)$ is unitary for every $A \in V$.

3. Characterization of the image. Let $A \in V$ have eigenvalues $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Then $f(A)$ is unitary and its eigenvalues are

$$\mu_k = \frac{\lambda_k + \sqrt{-1}}{\lambda_k - \sqrt{-1}}, \quad k = 1, \dots, n.$$

For real λ, $|\lambda + \sqrt{-1}| = |\lambda - \sqrt{-1}|$, so each μ_k lies on the unit circle. Moreover, $\mu_k = 1$ would require $\lambda_k + \sqrt{-1} = \lambda_k - \sqrt{-1}$, i.e., $\sqrt{-1} = -\sqrt{-1}$, impossible. Hence none of the eigenvalues equals 1.

Conversely, let $U \in \mathrm{U}(n)$ be a unitary matrix with no eigenvalue equal to 1. Then $U - I$ is invertible. Define

$$A = \sqrt{-1}\,(I + U)(I - U)^{-1}.$$

One checks that A is Hermitian and that $f(A) = U$ (direct computation or using the inverse Cayley transform). Thus the image of f is exactly the set of unitary matrices that do not have 1 as an eigenvalue.

4. Injectivity of f. Suppose $f(A) = f(B)$ for some $A, B \in V$. Let $U = f(A) = f(B)$. From the characterization in part 3, U does not have eigenvalue 1, so $I - U$ is invertible and we can recover A and B via the inverse Cayley transform:

$$A = \sqrt{-1}\,(I + U)(I - U)^{-1} = B.$$

Hence $A = B$, so f is injective. □